

# Fermion Bag Solitons in the Massive Gross-Neveu and Massive Nambu-Jona-Lasinio Models in $1 + 1$ Dimensions: Inverse Scattering Analysis

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**Abstract.** Formation of fermion bag solitons is an important paradigm in the theory of hadron structure. We report here on our non-perturbative analysis of this phenomenon in the  $1 + 1$  dimensional massive Gross-Neveu model, in the large  $N$  limit. Our main result is that the extremal static bag configurations are reflectionless, as in the massless Gross-Neveu model. Explicit formulas for the profiles and masses of these solitons are presented. We also present a particular type of self-consistent reflectionless solitons which arise in the massive Nambu-Jona-Lasinio models, in the large- $N$  limit.

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## 1. Introduction

An important dynamical mechanism, by which fundamental particles acquire masses, is through interactions with vacuum condensates. Thus, a massive particle may carve out around itself a spherical region [1] or a shell [2] in which the condensate is suppressed, thus reducing the effective mass of the particle at the expense of volume and gradient energy associated with the condensate. This picture has interesting phenomenological consequences [1, 3].

This dynamical distortion of the homogeneous vacuum condensate configuration, namely, formation of fermion bag solitons, was demonstrated explicitly by Dashen, Hasslacher and Neveu (DHN) [4] many years ago, who studied semiclassical bound states in the  $1 + 1$  dimensional Gross-Neveu (GN) model [5], using the inverse scattering method [6]. Following DHN, Shei [7] has applied the inverse scattering method to study solitons in the  $1 + 1$  dimensional Nambu-Jona-Lasinio (NJL) model [8] in the large- $N$  limit.

Fermion bags in the GN model were discussed in the literature several other times since the work of DHN, using alternative methods [9, 10, 11]. For a review on these and related matters (with an emphasis on the relativistic Hartree-Fock approximation) see [12]. For a more recent review of static fermion bags in the GN model (with an emphasis on reflectionless backgrounds and supersymmetric quantum mechanics) see [13]. The

large- $N$  semiclassical DHN spectrum of these fermion bags turns out to be essentially correct also for finite  $N$ , as analysis of the exact factorizable S-matrix of the GN model reveals [14].

A variational calculation of these effects in the  $1 + 1$  dimensional massive generalization of the Gross-Neveu model, which we will refer to as MGN, was carried in [15] a few years ago, and more recently in [16]. Very recently, we studied static fermion bags in the MGN model [17], which we obtained using the inverse-scattering formalism, thus avoiding the need to choose a trial variational field configuration. Our main result in [17] is that the extremal static bag configurations are reflectionless, as in the massless Gross-Neveu model. In the next section we briefly review the results of [17], leaving technical details out. Then, in Section 3, we show that a subclass of the reflectionless solitons of [17] arise self-consistently in the  $1 + 1$  dimensional massive NJL (MNJL) model. The latter extends the results of [7, 18] for the massless NJL model. Solitons in the MNJL model were also recently studied in [16], where a derivative expansion was carried out around a particular soliton background of the corresponding massless NJL model.

## 2. Solitons in the Massive Gross-Neveu Model

One way of writing the action for the MGN model is

$$S = \int d^2x \left\{ \sum_{a=1}^N \bar{\psi}_a [i\!\not{\partial} - \sigma] \psi_a - \frac{1}{2g^2} (\sigma^2 - 2M\sigma) \right\}, \quad (1)$$

where the  $\psi_a$  ( $a = 1, \dots, N$ ) are massive Dirac fermions and  $\sigma$  is an auxiliary field. Integrating the  $\sigma$  out results in an equivalent form of (1), with quartic fermion self-interactions.

An obvious symmetry of (1) with its  $N$  Dirac spinors is  $U(N)$ . Actually, (1) is symmetric under the larger group  $O(2N)$  [4] (see also Section 1 of [13]). The fact that the symmetry group of (1) is  $O(2N)$  rather than  $U(N)$  is related to the fact that it is invariant against charge-conjugation, like the massless GN model. Consequently, the energy eigenvalues of the Dirac equation associated with (1),  $[i\!\not{\partial} - \sigma(x)] \psi = 0$ , come in  $\pm\omega$  pairs.

As usual, the theory (1) can be rewritten with the help of the scalar flavor singlet auxiliary field  $\sigma(x)$ . Also as usual, we take the large  $N$  limit holding  $\lambda \equiv Ng^2$  fixed. Integrating out the fermions, we obtain the bare effective action

$$S[\sigma] = -\frac{1}{2g^2} \int d^2x (\sigma^2 - 2M\sigma) - iN \text{Tr} \log (i\!\not{\partial} - \sigma). \quad (2)$$

Noting that  $\gamma_5(i\!\not{\partial} - \sigma) = -(i\!\not{\partial} + \sigma)\gamma_5$ , we can rewrite the  $\text{Tr} \log(i\!\not{\partial} - \sigma)$  as  $\frac{1}{2} \text{Tr} \log [-(i\!\not{\partial} - \sigma)(i\!\not{\partial} + \sigma)]$ . In this paper we focus on static soliton configurations. If  $\sigma$  is time independent, the latter expression may be further simplified to  $\frac{T}{2} \int \frac{d\omega}{2\pi} [\text{Tr} \log(h_+ - \omega^2) + \text{Tr} \log(h_- - \omega^2)]$  where  $h_{\pm} \equiv -\partial_x^2 + \sigma^2 \pm \sigma'$ , and where  $T$  is an infra-red

temporal regulator. As it turns out, the two Schrödinger operators  $h_{\pm}$  are isospectral (see Appendix A of [13] and Section 2 of [11]) and thus we obtain

$$S[\sigma] = -\frac{1}{2g^2} \int d^2x (\sigma^2 - 2M\sigma) - iNT \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \text{Tr} \log(h_- - \omega^2). \quad (3)$$

In contrast to the standard massless GN model, the MGN model studied here is not invariant under the  $Z_2$  symmetry  $\psi \rightarrow \gamma_5 \psi$ ,  $\sigma \rightarrow -\sigma$ , and the physics is correspondingly quite different. As a result of the  $Z_2$  degeneracy of its vacuum, the GN model contains a soliton (the so called CCGZ kink [19, 4, 9, 11, 13]) in which the  $\sigma$  field takes on equal and opposite values at  $x = \pm\infty$ . The stability of this soliton is obviously guaranteed by topological considerations. With any non-zero  $M$  the vacuum value of  $\sigma$  is unique and the CCGZ kink becomes infinitely massive and disappears. If any soliton exists at all, its stability has to depend on the energetics of trapping fermions.

Let us briefly recall the computation of the unique vacuum of the MGN model. We shall follow [15]. For an earlier analysis of the MGN ground state (as well as its thermodynamics), see [20]. Setting  $\sigma$  to a constant we obtain from (3) the renormalized effective potential (per flavor)  $V(\sigma, \mu) = \frac{\sigma^2}{4\pi} \log \frac{\sigma^2}{e\mu^2} + \frac{1}{\lambda(\mu)} \left[ \frac{\sigma^2}{2} - M(\mu)\sigma \right]$ , where  $\mu$  is a sliding renormalization scale with  $\lambda(\mu) = Ng^2(\mu)$  and  $M(\mu)$  the running couplings. By equating the coefficient of  $\sigma^2$  in two versions of  $V$ , one defined with  $\mu_1$  and the other with  $\mu_2$ , we find immediately that  $\frac{1}{\lambda(\mu_1)} - \frac{1}{\lambda(\mu_2)} = \frac{1}{\pi} \log \frac{\mu_1}{\mu_2}$  and thus the coupling  $\lambda$  is asymptotically free, just as in the GN model. Furthermore, by equating the coefficient of  $\sigma$  in  $V$  we see that the ratio  $\frac{M(\mu)}{\lambda(\mu)}$  is a renormalization group invariant. Thus,  $M$  and  $\lambda$  have the same scale dependence.

Without loss of generality we assume that  $M(\mu) > 0$  and thus the absolute minimum of  $V(\sigma, \mu)$ , namely, the vacuum condensate  $m = \langle \sigma \rangle$ , is the unique (and positive) solution of the gap equation  $\left. \frac{dV}{d\sigma} \right|_{\sigma=m} = m \left[ \frac{1}{\pi} \log \frac{m}{\mu} + \frac{1}{\lambda(\mu)} \right] - \frac{M(\mu)}{\lambda(\mu)} = 0$ . Referring to (1), we see that  $m$  is the mass of the fermion. Using the explicit scale dependence of  $\lambda(\mu)$ , we can re-write the gap equation as  $\frac{m}{\lambda(m)} = \frac{M(\mu)}{\lambda(\mu)}$ , which shows manifestly that  $m$ , an observable physical quantity, is a renormalization group invariant. This equation also implies that  $M(m) = m$ , which makes sense physically.

Fermion bags correspond to inhomogeneous solutions of the saddle-point equation  $\frac{\delta S}{\delta \sigma(x,t)} = 0$ . In particular, static bags  $\sigma(x)$  are the extremal configurations of the energy functional (per flavor)  $\mathcal{E}[\sigma(x)] = -\frac{S[\sigma(x)]}{NT}$ , subjected to the boundary condition that  $\sigma(x)$  relaxes to its unique vacuum expectation value  $m$  at  $x = \pm\infty$ . More specifically, we have to evaluate the energy functional of a static configuration  $\sigma(x)$ , obeying the appropriate boundary conditions at spatial infinity, which supports  $K$  pairs of bound states of the Dirac equation at energies  $\pm\omega_n$ ,  $n = 1, \dots, K$  (where, of course,  $\omega_n^2 < m^2$ ). The bound states at  $\pm\omega_n$  are to be considered together, due to the charge conjugation invariance of the GN model. Due to Pauli's exclusion principle, we can populate each of the bound states  $\pm\omega_n$  with up to  $N$  fermions. In such a typical multiparticle state, the negative frequency state is populated by  $N - h_n$  fermions (i.e., by  $h_n$  holes or

antiparticles) and the positive frequency state contains  $p_n$  fermions (or particles). We shall refer to the total number of particles and antiparticles trapped in the  $n$ -th pair of bound states  $\nu_n = p_n + h_n$  as the valence, or occupation number of that pair.

The energy functional  $\mathcal{E}[\sigma(x)]$  is, in principle, a complicated and generally unknown functional of  $\sigma(x)$  and of its derivatives (which furthermore, requires regularization). Thus, the extremum condition  $\frac{\delta \mathcal{E}[\sigma]}{\delta \sigma(x)} = 0$ , as a functional equation for  $\sigma(x)$ , seems intractable. The considerable complexity of the functional equations that determine the extremal  $\sigma(x)$  configurations is the source of all difficulties that arise in any attempt to solve the model under consideration. DHN found a way around this difficulty in the case of the GN model [4]. They have used inverse scattering techniques [6] to express the (regulated) energy functional  $\mathcal{E}[\sigma]$  in terms of the so-called “scattering data” associated with, e.g., the hamiltonian  $h_-$  mentioned above (and thus with  $\sigma(x)$ ), and then solved the extremum condition  $\frac{\delta \mathcal{E}[\sigma]}{\delta \sigma(x)} = 0$  with respect to those data.

The scattering data associated with  $h_-$  are [6] the reflection amplitude  $r(k)$  of the Schrödinger operator  $h_-$  at momentum  $k$ , the number  $K$  of bound states in  $h_-$  and their corresponding energies  $0 < \omega_n^2 \leq m^2$ , ( $n = 1, \dots, K$ ), and also additional  $K$  parameters  $\{c_n\}$ , where  $c_n$  has to do with the normalization of the  $n$ th bound state wave function  $\psi_n$  of  $h_-$ . More precisely, the  $n$ th bound state wave function, with energy  $\omega_n^2$ , must decay as  $\psi_n(x) \sim \text{const.} \exp -\kappa_n x$  as  $x \rightarrow \infty$ , where  $0 < \kappa_n = \sqrt{m^2 - \omega_n^2}$ . If we impose that  $\psi_n(x)$  be normalized, this will determine the constant coefficient as  $c_n$ . (With no loss of generality, we may take  $c_n > 0$ .) Recall that  $r(-k) = r^*(k)$ , since the Schrödinger potential  $V(x) = \sigma^2(x) - \sigma'(x)$  is real. Thus, only the values of  $r(k)$  for  $k > 0$  enter the scattering data. The scattering data are independent variables, which determine  $V(x)$  uniquely, assuming  $V(x)$  belongs to a certain class of potentials which fall-off fast enough toward infinity. (Since the MGN does not bear topological solitons, neither  $h_-$  nor  $h_+$  can have a normalizable zero energy eigenstate. Thus, all the  $\omega_n$  are strictly positive.)

We can apply directly the results of DHN in order to write down that part of  $\mathcal{E}[\sigma(x)]$  which is common to the MGN and GN models, i.e.,  $\mathcal{E}[\sigma(x)]$  with its term proportional to  $M$  removed, in terms of the scattering data. For lack of space we shall not write DHN’s expression for the energy functional explicitly. Suffice it is to mention at this point that the “DHN-part” of  $\mathcal{E}[\sigma(x)]$  depends on the reflection amplitude only via certain regular dispersion integrals of the quantity  $\log[1 - |r(k)|^2]$ . The well-known reflectionless nature of solitons in the GN model is a direct consequence of this simple fact.

In order to complete the task of expressing the effective action of the MGN model in terms of the scattering data, we have to find such a representation for the remaining piece of  $\mathcal{E}[\sigma(x)]$  proportional to  $M$ , namely, for  $-\frac{M}{\lambda} \int_{-\infty}^{\infty} (\sigma(x) - m) dx$ . The latter integral cannot be expressed in terms of the scattering data based on the trace identities of the Schrödinger operator  $h_-$  discussed in Appendix B of [4]. Evidently, new analysis is required to obtain its representation in terms of the scattering data. Happily enough,

we were able to obtain such a representation in [17], which reads

$$\int_{-\infty}^{\infty} (\sigma(x) - m) dx = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log[1 - |r(k)|^2]}{im - k} dk + \sum_{n=1}^K \log \left( \frac{m - \kappa_n}{m + \kappa_n} \right). \quad (4)$$

Thus, the  $M$ -dependent part of  $\mathcal{E}[\sigma(x)]$ , like its “DHN-part”, depends on the reflection amplitude only via the combination  $\log[1 - |r(k)|^2]$ . Combining these two terms together, it follows that  $\delta\mathcal{E}^{reg}[\sigma(x)]/\delta r(k) = F(k) r^*(k)/(1 - |r(k)|^2)$ , where  $F(k)$  is a calculable function, which does not vanish identically. Thus,  $r(k) \equiv 0$  is the unique solution of the variational equation  $\delta\mathcal{E}^{reg}[\sigma(x)]/\delta r(k) = 0$ . Static extremal bags in the MGN model are *reflectionless*, as their counterparts in the GN model.

Explicit formulas for reflectionless  $\sigma(x)$  configurations with an arbitrary number  $K$  of pairs of bound states are displayed in Appendix B of [13]. In particular, the one which supports a single pair of bound states at energies  $\pm\omega_b$  ( $\kappa = \sqrt{m^2 - \omega_b^2}$ ), the one originally discovered by DHN, is

$$\sigma(x) = m + \kappa \left[ \tanh \left( \kappa x - \frac{1}{4} \log \frac{m + \kappa}{m - \kappa} \right) - \tanh \left( \kappa x + \frac{1}{4} \log \frac{m + \kappa}{m - \kappa} \right) \right]. \quad (5)$$

We see that the formidable problem of finding the extremal  $\sigma(x)$  configurations of the energy functional  $\mathcal{E}[\sigma]$  is reduced to the simpler problem of extremizing an ordinary function  $\mathcal{E}(\omega_n, c_n) = \mathcal{E}[\sigma(x; \omega_n, c_n)]$  with respect to the  $2K$  parameters  $\{c_n, \omega_n\}$  that determine the reflectionless background  $\sigma(x)$ . If we solve this ordinary extremum problem, we will be able to calculate the mass of the fermion bag. This we did in detail in [17]. Let us sketch the procedure and state the final result:

The bare regulated energy function  $\mathcal{E}(\omega_n)$  which depends on the bare couplings  $\lambda$  and  $M$  and on the UV-cutoff  $\Lambda$  explicitly can be renormalized, in a manner essentially similar to the effective potential, as was described above.  $\mathcal{E}$  is independent of the  $c_n$ ’s, which appear in the scattering data. (The latter are thus flat directions for the energy function and determine the collective coordinates of the soliton.) The renormalized energy function thus obtained is a sum of the form  $\sum_{n=1}^K f(\omega_n, \nu_n)$  where  $f(\omega, \nu)$  is a known function, which depends also on the physical mass  $m$  explicitly, and also through the RG-invariant ratio  $\gamma = \frac{1}{\lambda(m)} = \frac{M(\mu)}{m\lambda(\mu)}$ . Thus, the extremum condition fixes each  $\omega_n$  in terms of the number of the total number  $\nu_n$  of particles and holes trapped in the bound states of the Dirac equation at  $\pm\omega_n$ , and not by the numbers of trapped particles and holes separately (see (6)). This fact is a manifestation of the underlying  $O(2N)$  symmetry, which treats particles and holes symmetrically. Moreover, it indicates [17] that this pair of bound states gives rise to an  $O(2N)$  antisymmetric tensor multiplet of rank  $\nu_n$  of soliton states. (As it turns out, only tensors of ranks  $0 < \nu_n < N$  correspond to viable solitons [17].) The soliton as a whole is therefore the tensor product of all these antisymmetric tensor multiplets. Finally, we showed in [17] that only the irreducible ( $K = 1$ ) soliton was protected by energy conservation and  $O(2N)$  symmetry against decaying into lighter solitons (or free massive fermions). Its profile is given by (5), where  $\kappa = m \sin \theta$  (or, equivalently,  $\omega = m \cos \theta$ ), with  $0 < \theta < \pi/2$ , and where  $\theta$  is

determined by the extremum condition

$$\frac{\theta}{\pi} + \gamma \tan \theta = \frac{\nu}{2N}. \quad (6)$$

The left-hand side of (6) is a monotonically increasing function. Therefore, (6) has a unique solution in the interval  $[0, \pi/2]$ . This solution is evidently smaller than  $\theta^{\text{GN}} = \frac{\pi\nu}{2N}$ , the corresponding value of  $\theta$  in the GN model for the same occupation number. Thus, the corresponding bound state energy  $\omega = m \cos \theta$  in the MGN model is higher than its GN counterpart, and thus less bound. The soliton mass (i.e., the renormalized energy function, evaluated at the solution of (6)) is

$$\mathcal{M}(\nu) = Nm \left( \frac{2}{\pi} \sin \theta + \gamma \log \frac{1 + \sin \theta}{1 - \sin \theta} \right). \quad (7)$$

This coincide with the corresponding results of variational calculations presented in [15, 16], which were based on (5) as a trial configuration. In fact, it was realized in [16] that the trial configuration (5) is an exact solution of the extremum condition  $\frac{\delta \mathcal{E}[\sigma]}{\delta \sigma(x)} = 0$ , provided (6) is used to fix  $\kappa = m \sin \theta$ .

### 3. Reflectionless solitons in the Massive Nambu-Jona-Lasinio Model

It is natural to inquire whether the results of the previous section carry over to the phenomenologically interesting MNJL model. The action for the MNJL model may be written as a generalization of (1),  $S = \int d^2x \left\{ \sum_{a=1}^N \bar{\psi}_a [i\partial - (\sigma + i\pi\gamma_5)] \psi_a - \frac{1}{2g^2} (\sigma^2 + \pi^2 - 2M\sigma) \right\}$ , where  $\pi(x)$  is a pseudo-scalar auxiliary field. (Here we assumed that the  $2 \times 2$  chiral mass matrix does not have a pseudo-scalar component, but this does not restrict the generality of our discussion in any way. This particular orientation of the mass matrix can be always reached at by performing a global - and therefore, anomaly free- chiral rotation in the  $\sigma - \pi$  plane.)

As in our discussion of the MGN model, we can integrate out the fermions, and obtain the bare effective action

$$S[\sigma] = -\frac{1}{2g^2} \int d^2x (\sigma^2 + \pi^2 - 2M\sigma) - iN \text{Tr} \log (i\partial - \sigma - i\pi\gamma_5). \quad (8)$$

As before, we take the large  $N$  limit, holding  $\lambda \equiv Ng^2$  fixed. Unlike the NJL model, with its continuum of degenerate vacua, the ground state of the MNJL model (8) is unique, as in the MGN model. It corresponds to a constant field configuration, where  $\pi = 0$  and where  $\sigma = m$  is determined by an equation identical to the one which arises in the MGN model.

Shei [7] has studied static solitons in the NJL model (i.e.,  $M = 0$  in (8)) using inverse scattering techniques. Similarly to DHN's results for the GN model, he concluded that extremal soliton profiles are reflectionless. Some of his results were rederived in [18], using a certain method based on properties of the diagonal resolvent of the Dirac operator, (which was applied first to the GN model in [11]).

Could Shei's analysis be extended to study solitons in the MNJL model, similarly to the extension of DHN's inverse scattering analysis to the MGN model? Could it be that

the self-consistent static soliton backgrounds in the MNJL model are reflectionless? It seems that all we need in order to answer these questions is a generalization of (4) to the case in which the Dirac operator involves a pseudo-scalar background  $\pi(x)$ . Unfortunately, we were not able (so-far) to find such a generalization, and therefore we cannot answer these questions in general at the moment. However, we were able to find a particular family of self-consistent reflectionless static solitons in the MNJL model by making an educated guess, as we shall now explain.

The spectrum of the Dirac equation associated with (8) is *not* invariant against charge-conjugation, unless  $\pi(x) \equiv 0$ . Thus, the bound states corresponding to a static soliton background are not paired, in general. In particular, as has been shown by Shei, there exist solitons in the NJL model which bind fermions into a single bound state. However, he has also found solitons with charge-conjugation-invariant spectrum (see Eqs.(3.25)-(3.28) in [7]), with a pair of bound states  $\pm\omega_b$ , in which  $\pi(x) = 0$  identically, and  $\sigma(x)$  is given by (5), which thus coincide with the DHN solitons in the GN model, for which  $\omega_b = m \cos(\frac{\pi\nu}{2N})$  and  $\mathcal{M}_{\text{DHN}}(\nu) = \frac{2Nm}{\pi} \sin(\frac{\pi\nu}{2N})$ . However, unlike in the GN model, in the NJL model we must choose  $p = h = \frac{\nu}{2}$  (i.e., a soliton of this type must trap an equal number of fermions and anti-fermions). The reason for this restriction is not hard to understand physically: the total chiral rotation  $\Delta\theta$ , namely, the difference in  $\arctan \frac{\pi(x)}{\sigma(x)}$  between the two ends of the one dimensional space, must be related to the fermion number charge  $n_f$  deposited in the soliton according to  $\Delta\theta = -\frac{2\pi n_f}{N}$  [21] (see also Eqs. (5.10) and (5.22) in [18]). The soliton profile  $(\sigma(x), \pi(x))$  under consideration, starts at the vacuum point  $(m, 0)$  at  $x = -\infty$  and returns to it at  $x = +\infty$ . Thus,  $\Delta\theta = n_f = p - h = 0$  for this soliton.

Now, any static soliton profile in the MNJL model must start at the *unique* vacuum  $(m, 0)$  at  $x = -\infty$  and return to it at  $x = +\infty$ . Thus, it should bring about null total chiral rotation, precisely as Shei's charge-conjugation-invariant configuration does. Therefore, if the MNJL bears *reflectionless* static solitons, they must be of this form (or charge-conjugate invariant generalizations thereof, with more pairs of paired bound states). The only thing that should change compared to the NJL model is the quantization condition, relating  $\omega_b$  and  $\nu$ .

We have verified that this is indeed the case, simply by substituting this configuration into the static inhomogeneous saddle-point equations associated with (8). Varying (8) with respect to  $\pi(x)$  we *obviously* obtain an equation identical in form to that of the NJL model. (For the latter, see the second equation in (5.1) in [18]). Using the explicit expressions for the entries of the diagonal resolvent of the Dirac operator with a reflectionless  $(\sigma(x), \pi(x))$  background with two bound states (Eqs. (4.13) and (4.14) in [18] with paired bound states  $\omega_2 = -\omega_1$ ), we see that  $\pi(x) \equiv 0$  is indeed a solution of that equation. (Here, having  $p = h = \frac{\nu}{2}$  is essential.). This  $\pi$ -equation leaves  $\omega_1$  an undetermined function of  $\nu$ . We still have to vary with respect to  $\sigma(x)$ . Substituting the explicit expressions for the appropriate entries of the diagonal resolvent of the Dirac operator (Eqs.(4.13), (4.14) and (2.10) of [18]) in the saddle-point equation arising from variation with respect to  $\sigma(x)$ , and using the simplifying identity Eq. (2.24)

of [22], we arrive simply at the static saddle-point equation of the MGN model, which is solved by  $\sigma(x)$  given by (5) and the quantization condition (6), leading to soliton mass (7). Thus, a restricted subset of the extremal reflectionless solitons of the MGN model appear, not surprisingly, also in the MNJL model. For these solitons  $\pi(x) \equiv 0$ . The question whether these solitons exhaust all possibilities in the two-dimensional MNJL model remains open.

## References

- [1] T. D. Lee and G. Wick, Phys. Rev. D **9**, 2291 (1974); R. Friedberg, T.D. Lee and R. Sirlin, Phys. Rev. D **13**, 2739 (1976); R. Friedberg and T.D. Lee, Phys. Rev. D **15**, 1694 (1977), *ibid.* **16**, 1096 (1977); A. Chodos, R. Jaffe, K. Johnson, C. Thorn, and V. Weisskopf, Phys. Rev. D **9**, 3471 (1974).
- [2] W. A. Bardeen, M. S. Chanowitz, S. D. Drell, M. Weinstein and T. M. Yan, Phys. Rev. D **11**, 1094 (1975); M. Creutz, Phys. Rev. D **10**, 1749 (1974).
- [3] R. MacKenzie, F. Wilczek and A. Zee, Phys. Rev. Lett. **53**, 2203 (1984), and references therein.
- [4] R.F. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. D **12**, 2443 (1975).
- [5] D.J. Gross and A. Neveu, Phys. Rev. D **10**, 3235 (1974).
- [6] The literature on physical applications of inverse scattering theory is enormously vast. In the present paper we stick to the conventions of S. Novikov, S.V. Manakov, L.P. Pitaevsky and V.E. Zakharov, *Theory of Solitons - The Inverse Scattering Method* (Consultants Bureau, New York, 1984), (Contemporary Soviet Mathematics), which should be consulted for further references.
- [7] S.-S. Shei, Phys. Rev. D **14**, 535 (1976).
- [8] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961); **124**, 246 (1961).
- [9] See e.g., A. Klein, Phys. Rev. D **14**, 558 (1976); R. Pausch, M. Thies and V. L. Dolman, Z. Phys. A **338**, 441 (1991).
- [10] A. Neveu and N. Papanicolaou, Commun. Math. Phys. **58**, 31 (1978); N. Papanicolaou, Ann. Phys. **136**, 210 (1981); R. de Mello Koch, *The Large- $N$  Limit of Fermionic Systems*, PhD Thesis, Univ. of the Witwatersrand, 1997.
- [11] J. Feinberg, Phys. Rev. D **51**, 4503 (1995).
- [12] V. Schön and M. Thies, *2d Model Field Theories at Finite Temperature and Density*, in *At the Frontiers of Particle Physics - Handbook of QCD* (vol. 3), M. Shifman (Ed.), (World Scientific, Singapore, 2001.) (hep-th/0008175).
- [13] J. Feinberg, Ann. Phys. **309**, 166 (2004).
- [14] A.B. Zamolodchikov and Al.B. Zamolodchikov, Ann. Phys. **120**, 253 (1979); M. Karowski, H.J. Thun, Nucl. Phys. **B190**, 61 (1981); For a recent discussion see P. Fendley and H. Saleur, Phys. Rev. D **65**, 025001 (2002).
- [15] J. Feinberg and A. Zee, Phys. Lett. **B411**, 134 (1997).
- [16] M. Thies and K. Urlichs, Phys. Rev. D **71**, 105008 (2005).
- [17] J. Feinberg and S. Hillel, Phys. Rev. D (2005), to appear (hep-th/0509019).
- [18] J. Feinberg and A. Zee, Phys. Rev. D **56**, 5050 (1997).
- [19] C.G. Callan, S. Coleman, D.J. Gross and A. Zee, unpublished; D.J. Gross in *Methods in Field Theory*, R. Balian and J. Zinn-Justin (Eds.), Les-Houches session XXVIII 1975 (North Holland, Amsterdam, 1976).
- [20] K. G. Klimenko, Theor. Math. Phys. **75**, 487 (1988); Teor. Mat. Fiz. **75**, 226 (1988) (in Russian).
- [21] J. Goldstone and F. Wilczek, Phys. Rev. Lett. **47**, 986 (1981); See also Eq. (4.44) of R. Aviv and A. Zee, Phys. Rev. D. **5**, 2372 (1972).
- [22] J. Feinberg, J. Math. Phys. **43**, 3927 (2002).